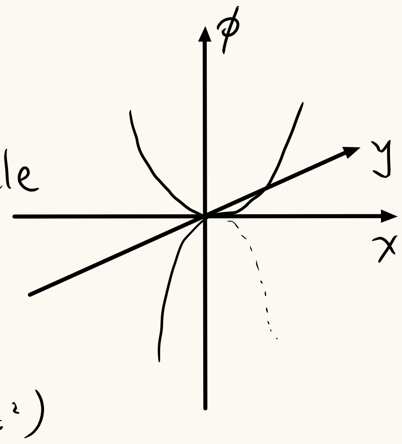
1. Paul trap / Radio-frequency trap



time-varying potential $\phi = \frac{V_0}{2r_0^2} \underbrace{\cos(\omega_{rf} t)}_{\text{dynamical trapping}} (x^2-y^2)$ ~ saddle

Generally in 3D, $\phi = \underbrace{\frac{V_0}{2r_0^2}(\alpha x^2+\beta y^2+\gamma z^2)}_{dc} + \underbrace{\frac{V_{rf}}{2r_0^2}\cos(\omega_{rf}t)(\alpha' x^2+\beta' y^2+\gamma' z^2)}_{rf}$

Laplace's equation $\Rightarrow \begin{cases} \alpha+\beta+\gamma=0 \\ \alpha'+\beta'+\gamma'=0 \end{cases}$

Choose $\alpha=\beta=\gamma=0$, and to choose a cylindrically symmetric trap, we need $\alpha'=\beta'=-2\gamma'=1$

$\Rightarrow \phi = \frac{V_{rf}}{2r_0^2}\cos(\omega_{rf}t)(x^2+y^2-2z^2) = \frac{V_{rf}}{2r_0^2}\cos(\omega_{rf}t)(r^2-2z^2)$

Cartesian Coordinate   Cylindrical Coordinate

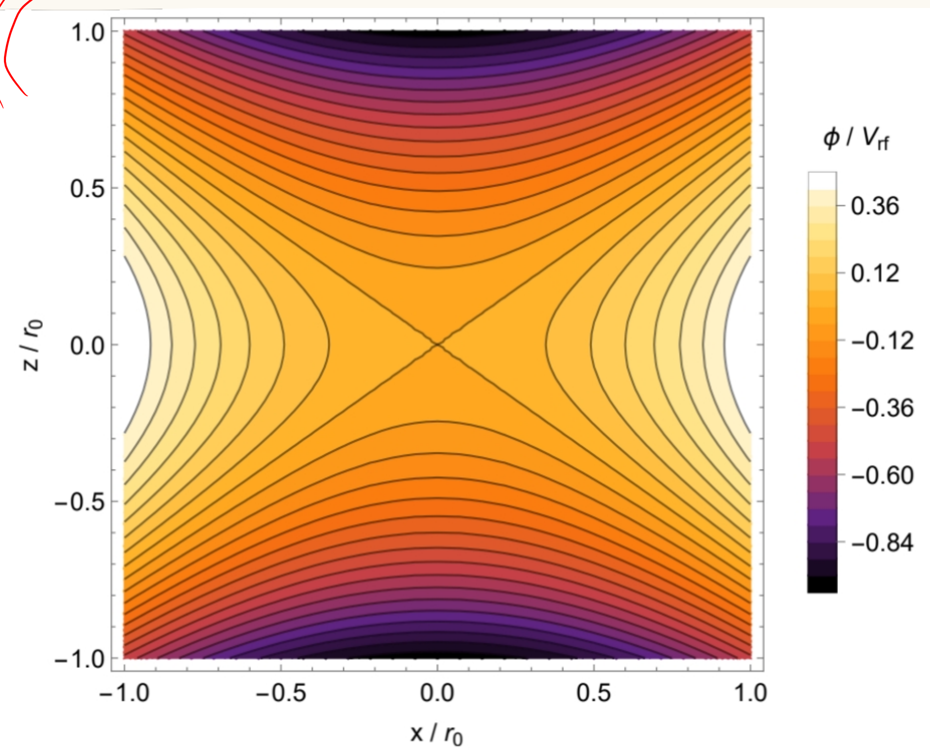
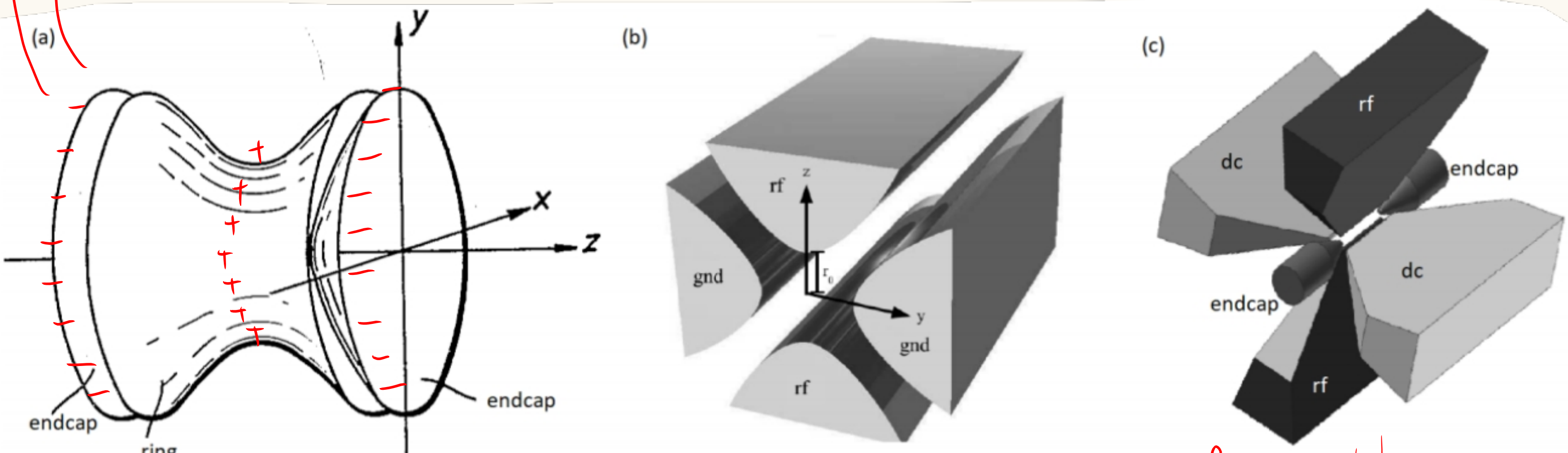


Figure 6.1: Electrostatic potential in the xz-plane for a 3D rf ion trap. The potential has cylindrical symmetry about the z-axis. The values of the contours correspond to Eq. (6.4) at $t = 0$.



$\begin{cases} \gamma = -\alpha-\beta \\ \alpha' = -\beta', \ \gamma'=0 \end{cases}$

Figure 6.2: Electrode geometries for making rf traps. (a) Cylindrically symmetric 3D rf trap. (b) Linear rf trap with no confinement along the long direction. (c) Linear rf trap with endcap electrodes for static confinement along the long direction.

used for trapping linear strings of ions

Consider an ion of charge e and mass m in a potential of the general form.
Consider the x-direction and set $\alpha = \alpha' = 1$, the equation of motion:

$$\frac{d^2x}{dt^2} = -\frac{e}{mr_0^2}\left[V_0 + V_{rf}\cos(\omega_{rf}t)\right]x$$

$\tau = \frac{\omega_{rf}t}{2}$, $a_x = \frac{4eV_0}{mr_0^2\omega_{rf}^2}$, $q_x = -\frac{2eV_{rf}}{mr_0^2\omega_{rf}^2}$ $\Longrightarrow$ $$\frac{d^2x}{d\tau^2} + \left[a_x - 2q_x\cos(2\tau)\right]x = 0$$ Mathieu equation

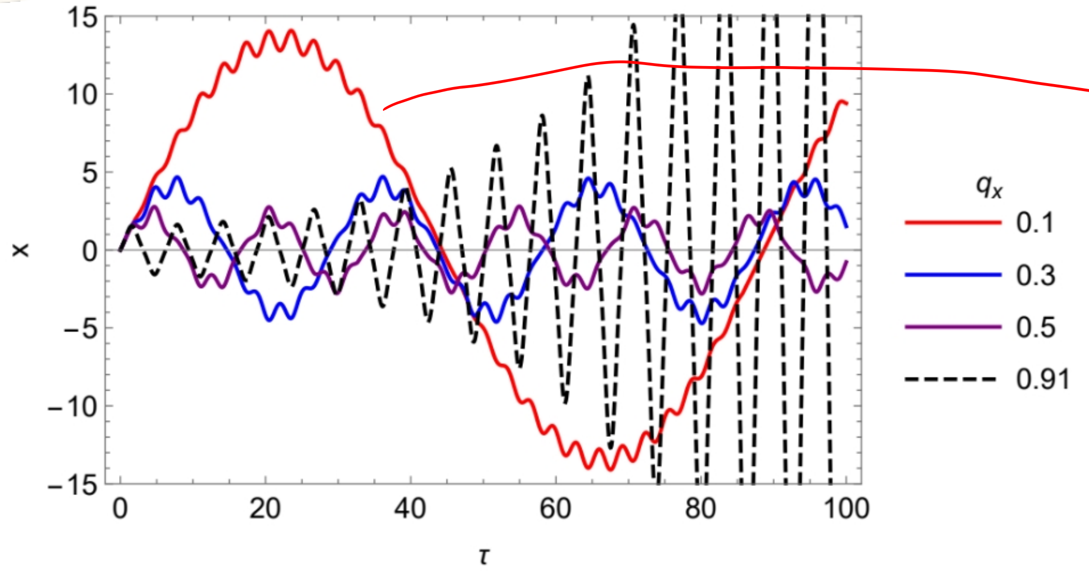


Figure 6.3: Solutions to the Mathieu equation, with $\underbrace{a_x = 0}_{\text{no dc potential}}$ and various values of q_x. The motion becomes unstable for $q_x > 0.9$ (black dashed line).

micromotion: a small amplitude oscillation at the rf frequency

$\updownarrow$ superimpose

secular motion: a slower and larger amplitude oscillation

$q_x \nearrow$, the frequencies of the two oscillations approach one another

The behavior at small q_x suggests $x = x_f + x_s$ (fast and slow oscillation), we assume $x_f \ll x_s$, $dx_f/dt \gg dx_s/dt$

$$\Longrightarrow \frac{d^2x_f}{d\tau^2} = 2q_x\cos(2\tau)\,x_s$$

Takes x_s to be a constant over the short period of high-frequency oscillation, so that

$$x_f = -\frac{q_x x_s}{2}\cos(2\tau)$$

$$\Longrightarrow \frac{d^2(x_s+x_f)}{d\tau^2} = 2q_x\cos(2\tau)\,(x_s+x_f)$$

$$\frac{d^2x_s}{d\tau^2} + \underbrace{\frac{d^2x_f}{d\tau^2}}_{\to 0} = 2q_x x_s \underbrace{\cos(2\tau)}_{\to 0} - q_x^2 x_s \underbrace{\cos^2(2\tau)}_{\to \frac{1}{2}}$$

time average over one period of the rf drive

$$\Longrightarrow \frac{d^2x_s}{d\tau^2} = -\frac{q_x^2}{2}x_s$$

$$\Longrightarrow \frac{d^2x_s}{dt^2} = -\left(\frac{e^2V_{rf}^2}{2m^2r_0^4\omega_{rf}^2}\right)x_s$$

slow motion is that of a harmonic oscillator of angular frequency $\omega_x = \frac{eV_{rf}}{\sqrt{2}mr_0^2\omega_{rf}}$

$\omega_z = 2\omega_x = 2\omega_y$ for a cylindrical trap

- macroscopic ion traps ~ trap frequencies ~ 1MHz
- microscopic ion traps formed on chips ~ trap frequencies ~ 10–100 MHz

2. Penning trap

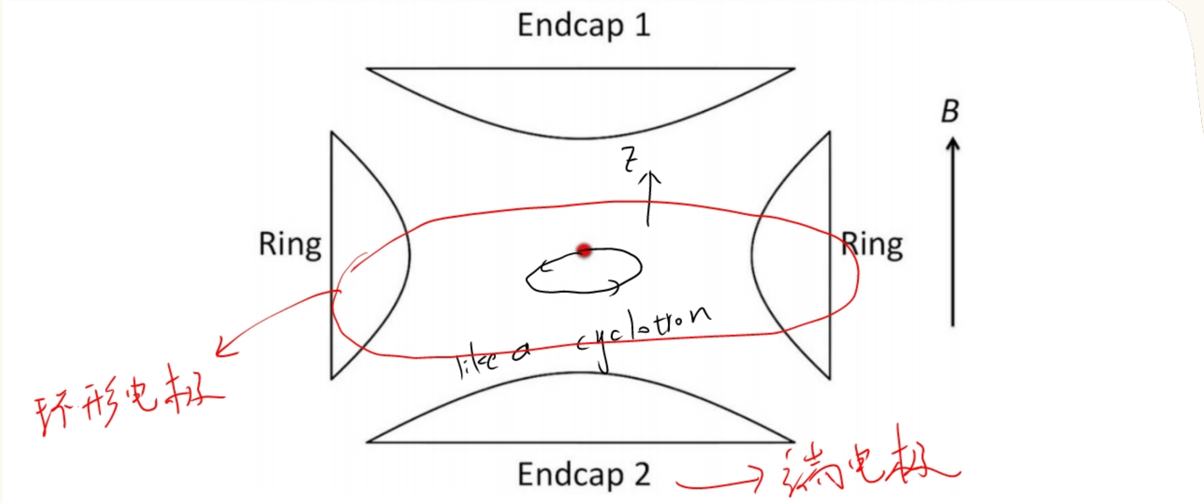
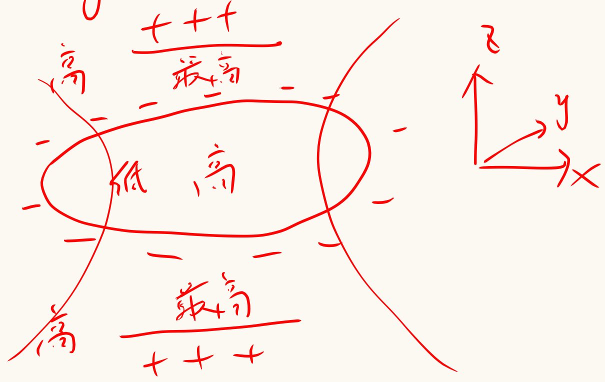


Figure 6.5: Cross section through the electrodes of a Penning trap. The trap has cylindrical symmetry about the vertical axis. The two endcaps are electrically connected and a dc voltage is applied between the ring and the endcaps.

$z \nearrow \quad \phi \nearrow$

$x, y \nearrow \quad \phi \searrow$



$$\phi = \frac{V_0}{2r_0^2}\left(z^2 - \frac{1}{2}(x^2+y^2)\right)$$

$$\begin{cases} \vec{E} = -\nabla\phi = -\frac{V_0}{r_0^2}\left(\vec{z} - \frac{1}{2}(\vec{x}+\vec{y})\right) \\ \vec{B} = B\vec{z} \\ \vec{F} = e(\vec{E} + v\times\vec{B}) \end{cases} \Longrightarrow \begin{cases} \ddot{x} = \frac{1}{2}\omega_z^2 x + \omega_c \dot{y} \\ \ddot{y} = \frac{1}{2}\omega_z^2 y - \omega_c \dot{x} \\ \ddot{z} = -\omega_z^2 z \end{cases}$$

cyclotron frequency $\omega_c = \frac{eB}{m}$

axial frequency $\omega_z = \sqrt{\frac{eV_0}{mr_0^2}}$

$\Downarrow$ $u = x+iy$

$$\ddot{u} = \frac{1}{2}\omega_z^2 u - i\omega_c\dot{u}$$

$\Downarrow$ solution of the form $u = Ae^{-i\omega t}$

$$\omega^2 - \omega_c\omega + \frac{1}{2}\omega_z = 0$$

$$\Downarrow$$

$$\begin{cases} \omega_+ = \frac{1}{2}\left(\omega_c + \sqrt{\omega_c^2 - 2\omega_z^2}\right) & \text{modified cyclotron frequency} \\ \omega_- = \frac{1}{2}\left(\omega_c - \sqrt{\omega_c^2 - 2\omega_z^2}\right) & \text{magnetron frequency } \omega_m \end{cases}$$

$$\omega_+ \gg \omega_z \gg \omega_-$$

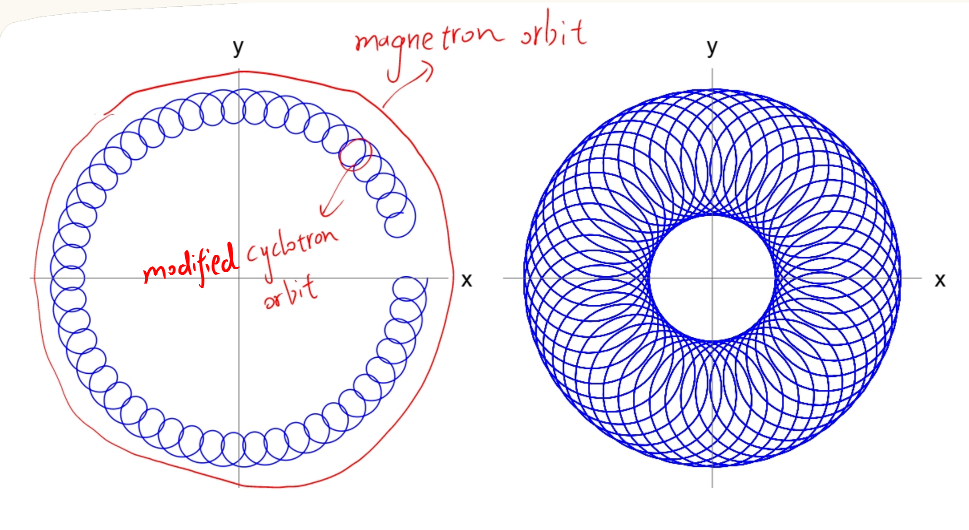


Figure 6.6: Example trajectories in the radial plane of an ion in a Penning trap. Small circles are cyclotron orbits, the larger one is the magnetron orbit.

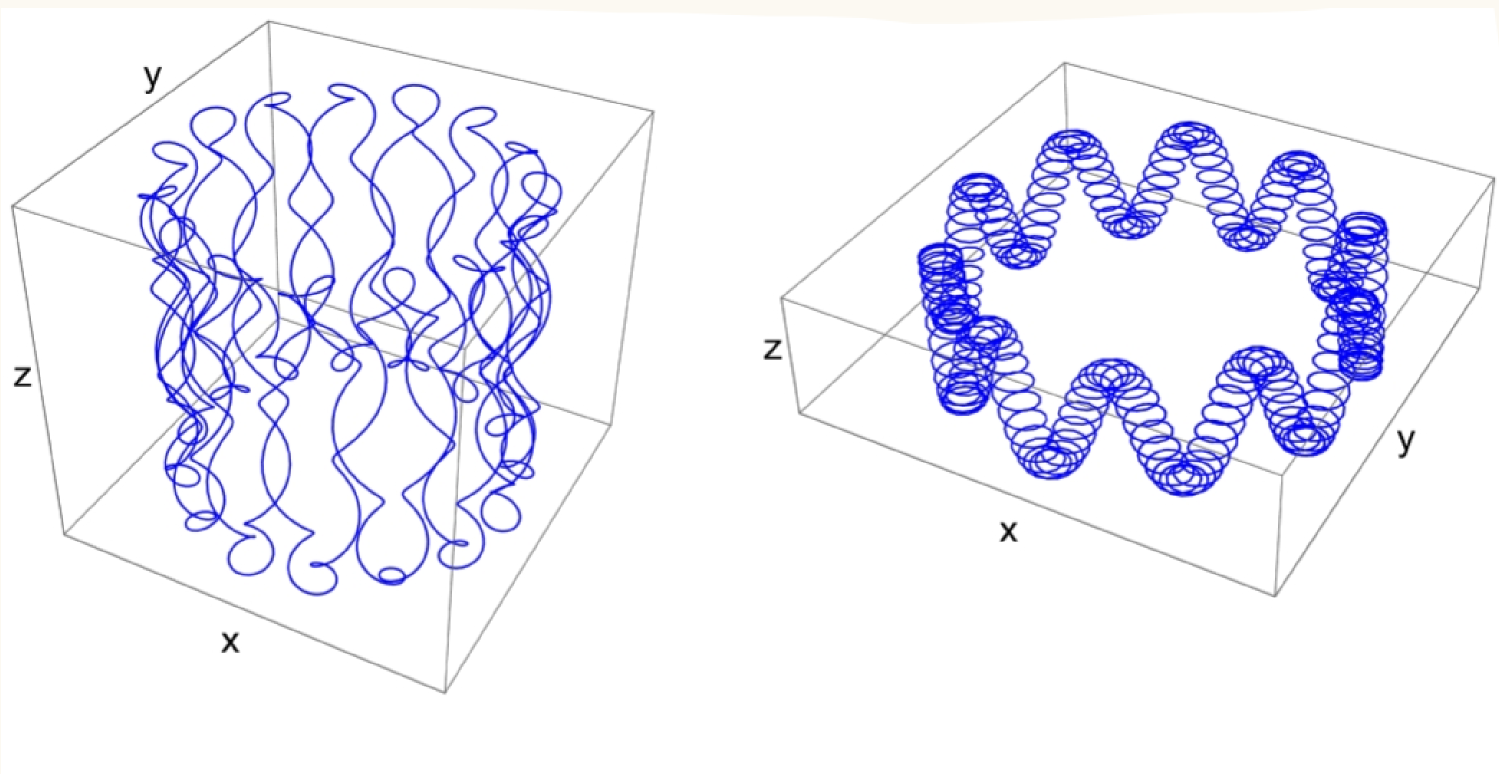


Figure 6.7: Examples of three-dimensional trajectories of an ion in a Penning trap.

3. Sideband cooling

4. Trapped ion quantum computing

Advantages
- Long trap lifetimes
- Long internal-state coherence times
- strong ion-ion interactions
- the existence of cycling transitions that can be used for coding and qubit readout

① Optical qubits : $^2S_{1/2}$ ground state $|1\rangle$ and $^2D_{5/2}$ excited state $|2\rangle$ ($Be^+, Ca^+, Mg^+, Sr^+, Ba^+$)

e.g. $^{40}Ca^+$

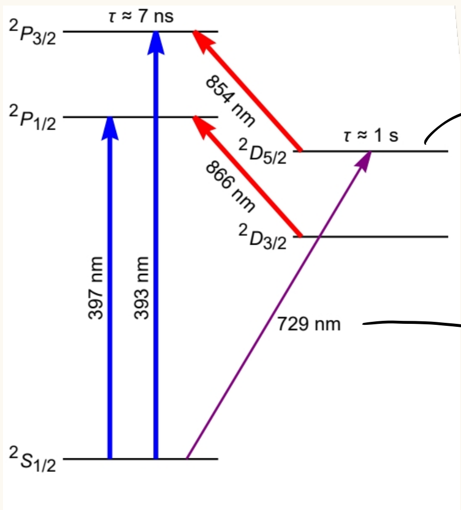


Figure 6.9: Energy levels and transitions of $^{40}Ca^+$.

→ metastable with a long lifetime

→ laser pulses at 729nm used to drive single qubit gate

Readout : turn on the laser cooling light and detect the fluorescence of the ion
- in $|1\rangle$ → fluoresces
- in $|2\rangle$ → does not ...

② Hyperfine qubits ~ infinite lifetimes

e.g. $^{43}Ca^+$, nuclear spin $I = 7/2$, $F=3$ and $F=4$ hyperfine states

$^2S_{1/2}$, $F = |\frac{1}{2} \pm \frac{7}{2}|$

total angular momentum

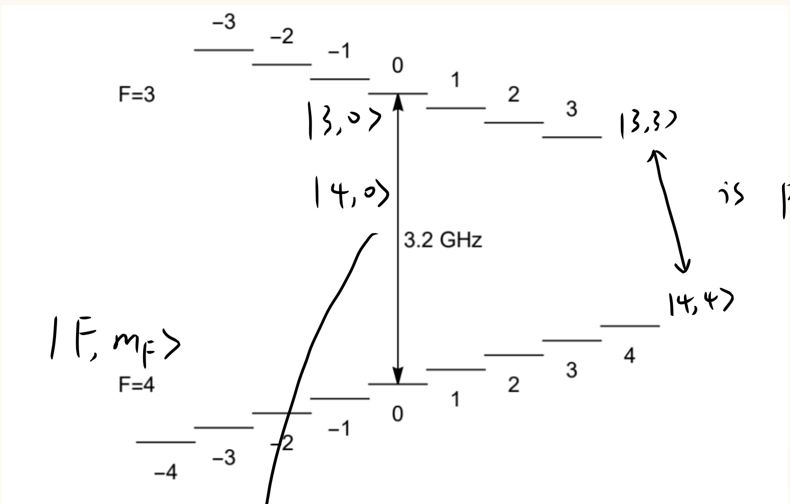
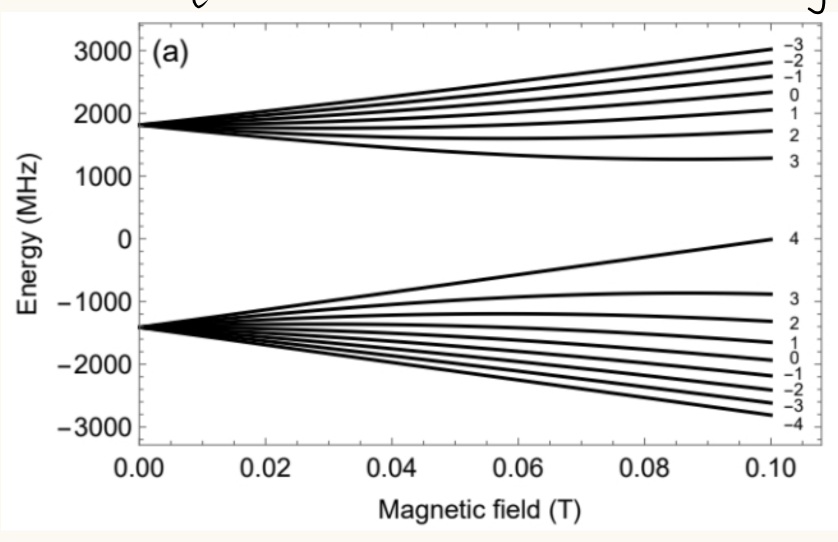


Figure 6.14: Hyperfine levels of $^{43}Ca^+$. The Zeeman splitting in a magnetic field is shown.

is prone to dephasing due to frequency fluctuations caused by magnetic field fluctuations (sensitivity 24.5 GHz/T)

a better choice : have no first-order Zeeman shift at zero field

In practice, a small magnetic field needs to be applied in order to lift the degeneracy, but this qubit still has a small magnetic field sensitivity.



We want to find a different choice of qubit gives zero sensitivity to magnetic field at a particular non-zero field, i.e $df/dB = 0$ Clock states

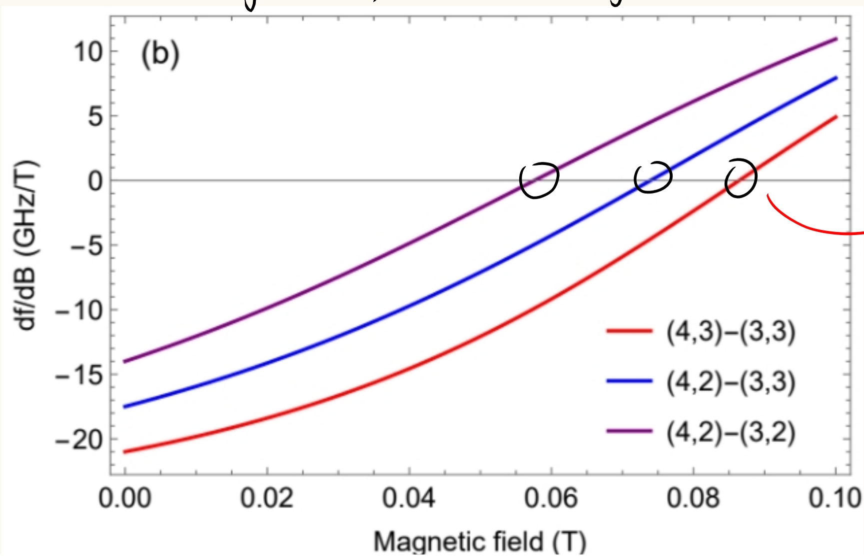


Figure 6.15: (a) Zeeman splitting of the ground state hyperfine components of $^{43}Ca^{+}$. (b) Sensitivity of qubit frequency to magnetic field, df/dB, as a function of magnetic field, for a few different qubit choices.

Readout: based on a hyperfine-selective cycling transition.

e.g. For $^{171}Yb^{+}$ $\quad {}^{2}S_{1/2} \rightrightarrows |0\rangle = |F=0, m_F=0\rangle$

$|1\rangle = |F=1, m_F=0\rangle$

hyperfine-selective cycling transition: $|F=1\rangle \xleftrightarrow{\text{Bright state}} |{}^{2}P_{1/2}, F=0\rangle$

$|F=0\rangle \xrightarrow{\times}$ Dark state

Here, $|0\rangle = |{}^{2}S_{1/2}, F=4\rangle \xrightarrow{\text{shelving}} {}^{2}D_{5/2}$

$|1\rangle = |{}^{2}S_{1/2}, F=3\rangle$

$\Downarrow$

cycling transition: ${}^{2}S_{1/2} \underset{397nm}{\longleftrightarrow} {}^{2}P_{1/2}$

Bright state: $|1\rangle$

Dark state: $|0\rangle$ (Because of shelving)

Once the ion reaches ${}^{2}D_{5/2}$, it remains there for a long time, because the state has such a long lifetime.

This is called "shelving"

③ Single-qubit gates

The dynamics here are simply that of a two-level atom interacting with light. The light is resonant with the transition. ($\delta = 0$)

light: $\vec{E} = \vec{E}_0 \cos(\omega t + \phi)$

$$\begin{cases} i\frac{dc_1}{dt} = \frac{\Omega}{2} e^{i\phi} c_2 \\ i\frac{dc_2}{dt} = \frac{\Omega}{2} e^{-i\phi} c_1 \end{cases} \Rightarrow \begin{cases} c_1(t) = \cos(\Omega t/2) c_1(0) - ie^{i\phi} \sin(\Omega t/2) c_2(0) \\ c_2(t) = -ie^{-i\phi} \sin(\Omega t/2) c_1(0) + \cos(\Omega t/2) c_2(0) \end{cases}$$

$$c_1(t)|1\rangle + c_2(t)|2\rangle = \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = U(t) \begin{pmatrix} c_1(0) \\ c_2(0) \end{pmatrix}, \quad U(t) = \begin{pmatrix} \cos(\Omega t/2) & -ie^{i\phi} \sin(\Omega t/2) \\ -ie^{-i\phi} \sin(\Omega t/2) & \cos(\Omega t/2) \end{pmatrix}$$

When $\phi = 0$

- $\Omega t = \pi/2$ ($\frac{\pi}{2}$ pulse) $\quad c_1(t) = \frac{1}{\sqrt{2}}(c_1(0) - ic_2(0))$, $c_2(t) = \frac{1}{\sqrt{2}}(-ic_1(0) + c_2(0))$ $\sim$ $|1\rangle \to \frac{1}{\sqrt{2}}(|1\rangle - i|2\rangle)$, $|2\rangle \to \frac{1}{\sqrt{2}}(-i|1\rangle + |2\rangle)$
- $\Omega t = \pi$ (π pulse) $\quad |1\rangle \to -i|2\rangle$, $|2\rangle \to -i|1\rangle$
- $\Omega t = 2\pi$ (2π pulse) $\quad |1\rangle \to -|1\rangle$, $|2\rangle \to -|2\rangle$

For an optical qubit, the transitions are driven using laser pulses.

For a hyperfine qubit (two states have the same parity):

single qubit manipulations
- be driven directly in the microwave domain
  (a magnetic dipole transition)
  long wavelength → difficult to address the individual ions ??
  using small microwave transmission lines microfabricated into the chip
- using a Raman transition in the optical domain (a two-photon process)

two lasers: ω_1, ω_2

$\delta = \omega_1 - \omega_2 - \omega_{12}$

$\Delta \gg \Omega_1, \Omega_2, \Gamma$

decay at rate Γ

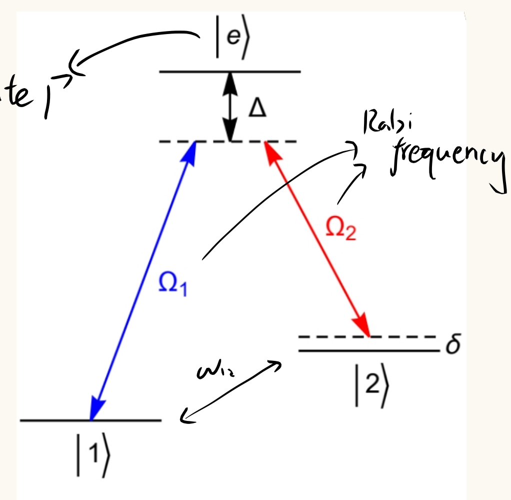


Figure 6.16: Driving a Raman transition

The dynamics are identical to that of a two-level system driven by a single frequency of light.

effective Rabi frequency $\Omega_{eff} = \frac{\Omega_1 \Omega_2}{2\Delta}$

effective two-photon detuning $\delta_{eff} = \delta - \frac{\Omega_1^2 - \Omega_2^2}{4\Delta}$

④ Two-qubit gates (fidelities 99.9% ; gate speeds 100 μs)

→ single ion axial trapping frequency

normal modes of motion in the axial direction
- Centre of Mass (COM) at ω_z
  oscillate in the same phase and with the same amplitude
- a stretch or breathing motion at $\sqrt{3}\omega_z$ (for two-ion chain)
  oscillate in opposite directions, the ions closer to centre move less than those near the edges

Choose the breathing mode, assume in the motional ground state

ions : 1, 2

internal states
- qubit states $|g\rangle, |e\rangle$
- auxiliary state $|a\rangle$

motional state : $|n\rangle$

Cirac-Zoller gate (Controlled-Z gate)

π-pulse on the red motional sideband of ion 1 ; 2π-pulse on the red motional sideband of ion 2 ?

$$|g\rangle_1 |g\rangle_2 |0\rangle_m \longrightarrow |g\rangle_1 |g\rangle_2 |0\rangle_m \longrightarrow |g\rangle_1 |g\rangle_2 |0\rangle_m$$

$$|g\rangle_1 |e\rangle_2 |0\rangle_m \longrightarrow |g\rangle_1 |e\rangle_2 |0\rangle_m \longrightarrow |g\rangle_1 |e\rangle_2 |0\rangle_m$$

$$|e\rangle_1 |g\rangle_2 |0\rangle_m \longrightarrow -i|g\rangle_1 |g\rangle_2 |1\rangle_m \longrightarrow i|g\rangle_1 |g\rangle_2 |1\rangle_m$$

why not → $|e\rangle_2 |0\rangle_m$

$$|e\rangle_1 |e\rangle_2 |0\rangle_m \longrightarrow -i|g\rangle_1 |e\rangle_2 |1\rangle_m \longrightarrow -i|g\rangle_1 |e\rangle_2 |1\rangle_m$$

$$\begin{cases} \omega = \omega_0 - \omega_t \\ t = \pi/\Omega \end{cases} \qquad \begin{cases} \omega = \omega_0 - \omega_t \\ t = 2\pi/\Omega \end{cases}$$

π-pulse on the red motional sideband of ion 1

$$\longrightarrow |g\rangle_1 |g\rangle_2 |0\rangle_m$$

$$\longrightarrow |g\rangle_1 |e\rangle_2 |0\rangle_m$$

$$\longrightarrow |e\rangle_1 |g\rangle_2 |0\rangle_m$$

$$\longrightarrow -|e\rangle_1 |e\rangle_2 |0\rangle_m$$

$\sim$

$$|g\rangle_1 |g\rangle_2 \longrightarrow |g\rangle_1 |g\rangle_2$$

$$|g\rangle_1 |e\rangle_2 \longrightarrow |g\rangle_1 |e\rangle_2$$

$$|e\rangle_1 |g\rangle_2 \longrightarrow |e\rangle_1 |g\rangle_2$$

$$|e\rangle_1 |e\rangle_2 \longrightarrow -|e\rangle_1 |e\rangle_2$$

Drawbacks
① requires ions to be cooled to the ground state of their collective motion
② requires individual addressing of each ion

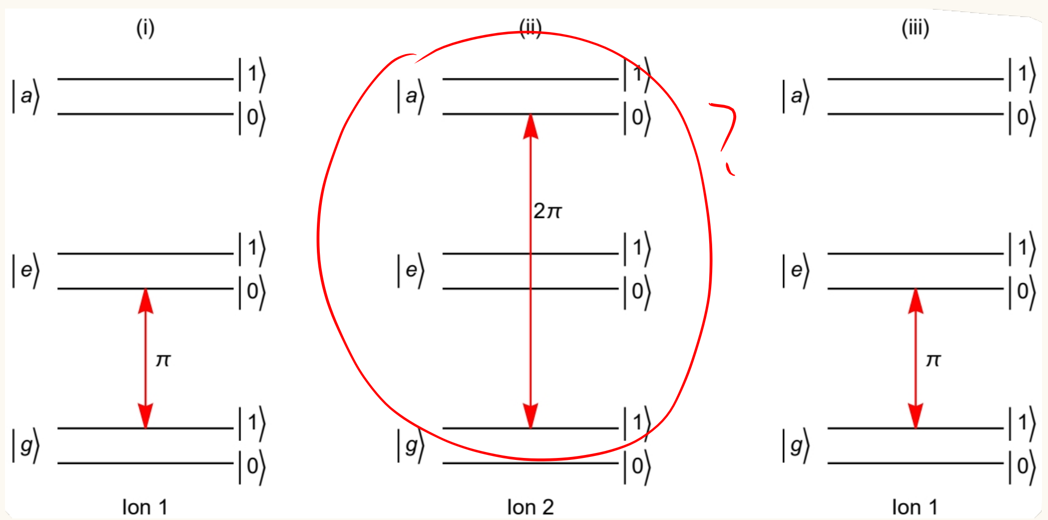


Figure 6.17: Implementation of a controlled-Z gate between a pair of ion qubits. $|g\rangle$ and $|e\rangle$ are the qubit states, and $|a\rangle$ is an auxiliary state. $|0\rangle$ and $|1\rangle$ refer to the (shared) motional state. The steps are applied in sequence. (i) The state of qubit 1 is mapped to the motional state. (ii) A π phase shift is applied to the motional state conditional on the state of qubit 2. (iii) The motional state is mapped back to ion 1.